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ORIGINAL ARTICLE

Fuzzy k-Primary Decomposition of Fuzzy k-Ideal in a Semiring

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Abstract In this paper, we establish that the Lasker-Noether theorem for a commutative ring may be generalized for a commutative semiring. We produce an example of an ideal in a Noetherian semiring which cannot be expressed as finite intersection of primary ideals of that semiring. But we manifest that if we consider an arbitrary k-ideal of a commutative Noetherian semiring, then it can be decomposed as finite intersection of primary k-ideals. Focus mainly on the fuzzy version of the above result, we are able to prove that in a commutative Noetherian semiring, every fuzzy k-ideal can be decomposed uniquely as finite intersection of fuzzy k-primary ideals of that semiring.

Keywords Semiring · k-ideal · Fuzzy primary ideals · Fuzzy k-primary ideal · Fuzzy k-primary decomposition

1. Introduction

Ideal theory is one of the essential tool for the study of structure of rings. In case of semirings, k-ideal theory has a prominent role for the study of structure of some class of semirings. Primary decomposition theorem plays a significant role in commutative ring theory. It can be considered as one kind of extension of fundamental theorem

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of arithmetic. Primary decomposition theorem for rings is also known as Lasker-Noether theorem which states that in a commutative Noetherian ring, every ideal can be represented as an intersection of finitely many primary ideals. Primary ideals, the generalization of prime ideals, may be considered as some power of prime ideals, but power of prime ideals are not necessarily primary ideals. After the introduction of primary decomposition of an ideal in Noetherian ring, some researchers started working on primary decomposition of an ideal in some other algebraic structures. J. N. Chaudhary and V. Gupta [3] worked on weak primary decomposition theorem in right Noetherian semiring. J. R. Mosher [19] considered the primary decomposition in a hemiring. R. E. Atani and S. E. Atani [1] investigated the primary ideals and irreducible ideals in a Noetherian semiring. Also, see [4].

Now, fuzzy ideal is one of the fundamental tool for the study of structure of rings and fuzzy k-ideal has a prominent role for the study of different class of semirings. The authors [13] have already studied some k-regularities of semirings in terms of fuzzy k-ideals of semirings. Like ring theory, fuzzy primary decomposition for semirings would be of interest for the representation of a fuzzy ideal in terms of fuzzy primary ideals of a semiring. In 1991, Kumar ([14, 15]) initiated the notion of fuzzy primary ideals and fuzzy irreducible ideals in a ring. Also, Malik and Mordeson [17] studied the notion of fuzzy primary ideals in a ring in the same year. He found that in a Noetherian ring, every fuzzy ideal can be represented as an intersection of finite number of fuzzy irreducible ideals. He also evinced the primality of every fuzzy irreducible ideal in a Noetherian ring. As a byproduct of this result, he acquired that in a Noetherian ring, every fuzzy ideal can be represented as a finite intersection of fuzzy primary ideals. Malik and Mordeson [18] also considered the primary decomposition of a fuzzy ideal in a Noetherian ring. It is quite natural to catechize whether the primary decomposition is possible for an arbitrary ideal in a Noetherian semiring. We uphold that in an arbitrary Noetherian semiring, an ideal may not be expressed as finite intersection of primary ideals. So, there do exist some Noetherian semirings in which primary decomposition is not possible for certain ideals. For example, we consider the matrix semiring $M_2(\mathbb{Z}_0^+)$ over the semiring $\mathbb{Z}_0^+$ of non-negative integers with respect to usual addition and multiplication of integers. The ideal $M_2(I)$ is not a k-ideal of $M_2(\mathbb{Z}_0^+)$ where $I = <2, 3>$. It can be readily checked that $M_2(I)$ cannot be expressed as finite intersection of primary ideals of $M_2(\mathbb{Z}_0^+)$. We explore that this decomposition is possible if we take k-ideals in spite of ideals in a semiring. For this reason, we consider the primary decomposition of a fuzzy k-ideal in a Noetherian semiring.

To set up our main desired result, we start with introducing the concept of fuzzy k-primary ideal of a semiring. Afterwards, we characterize fuzzy k-primary ideal of a semiring in various ways. Then, we introduce the notion of fuzzy k-irreducible ideal of a semiring. We prove that in a commutative Noetherian semiring, every fuzzy k-irreducible ideal is also a fuzzy k-primary ideal. Finally, we build up primary decomposition theorem for a fuzzy k-ideal of a semiring.

2. Preliminaries and Prerequisites

Firstly, we recall some definitions and results of semirings and fuzzy algebra which will be required to develop our paper. A non-empty set S together with two binary operations '+' and '$\cdot$' is said to be a semiring if (i) $(S,+)$ is an Abelian semigroup; (ii) $(S,\cdot)$ is a semigroup and (iii) $a\cdot(b+c)=a\cdot b+a\cdot c$ and $(b+c)\cdot a=b\cdot a+c\cdot a$ for all $a,b,c\in S$. The rudimentary results of semiring theory can be found in the books [9] and [11]. Throughout this paper we consider a commutative semiring S with zero element '0' and identity element '1'. It is well-known that in ring theory every ideal can be represented as the kernel of some ring homomorphism. But this result is not true for an ideal of a semiring, in general. Certainly, in semiring theory, there exists a class of ideals which can be represented as the kernel of some semiring homomorphism. In 1958, M. Henriksen [12] first introduced the notion of k-ideals (perhaps, the term 'k' stands for 'kernel') in semiring theory. An ideal I of a semiring S is said to be a k-ideal of S if for any $x\in S$ and $y\in I$, $x+y\in I\Longrightarrow x\in I$. Let A be a non-empty subset of a semiring S. Then, the k-closure of A, denoted by $\overline{A}$, is defined as : $\overline{A}=\{a\in S\mid a+b=c \text{ for some } b,c\in A\}$. In [10], it is shown that for any two non-empty subsets A,B of a semiring S, (i) $A\subseteq\overline{A}$; (ii) $A\subseteq B\Longrightarrow\overline{A}\subseteq\overline{B}$; (iii) $\overline{\overline{A}}=\overline{A}$; (iv) $\overline{AB}=\overline{\overline{A}\,\overline{B}}$.

Now, suppose that I is an ideal of a semiring S. Then, I is a k-ideal of S if and only if $\overline{I}=I$. Again, if I and J are two k-ideals of a semiring S, then $I\cap J$ is also a k-ideal of S. Let X be a non-empty set. A mapping $\mu: X\to[0,1]$ is called a fuzzy subset [20] of X. If μ_1 and μ_2 are two fuzzy subsets of a semiring S, then the composition of these two fuzzy subsets is a fuzzy subset of S, defined by: $(\mu_1\circ\mu_2)(x)=\sup\left\{\min\left(\mu_1(a),\mu_2(b)\right)\mid x=ab \text{ for some } a,b\in S\right\}$. Let μ be a fuzzy subset of a set X. For $t\in(0,1]$, the subset $\mu_t=\{x\in X\mid\mu(x)\geq t\}$ is called the level subset of μ. A fuzzy subset μ of a semiring S is a fuzzy ideal of S if and only if μ_t's are ideals of S for all $t\in \mathrm{Im}\mu$. These ideals are called level ideals of μ. It is also true that $\mu_{t_1}\subseteq\mu_{t_2}$ for $t_1>t_2$. A non-empty fuzzy subset μ of a semiring S (i.e., $\mu(x)\neq 0$ for some $x\in S$) is said to be fuzzy ideal of S if (i) $\mu(x+y)\geq\min(\mu(x),\mu(y))$ and (ii) $\mu(xy)\geq\max(\mu(x),\mu(y))$ for all $x,y\in S$. A fuzzy ideal μ of a semiring S is said to be a fuzzy k-ideal of S if $\mu(x)\geq\min(\mu(x+y),\mu(y))$ for any $x,y\in S$. Some basic definitions and rudimentary results of fuzzy ideals of semirings can be found in [2].

3. Fuzzy k-Primary Ideals of a Semiring

Before going to study fuzzy k-primary ideals of a semiring, we first introduce the definition of prime, k-prime and k-primary ideal of a semiring.

Definition 3.1 [9] *A proper ideal P of a semiring S is said to be prime if $AB\subseteq P$ for any two ideals A,B of S implies that either $A\subseteq P$ or $B\subseteq P$.*

A prime ideal of S is said to be a k-prime ideal of S if it is also a k-ideal of S.

Definition 3.2 *A proper ideal I of a semiring S is said to be a primary ideal of S if for any $x,y\in S$, $xy\in I$ implies that $x\in I$ or $y^n\in I$ for some $n\in\mathbb{Z}^+$.*

A primary ideal of S is said to be a k-primary ideal of S if it is also a k-ideal of S.

Example 3.1 (i) In the semiring $\mathbb{Z}_0^+$ of non-negative integers, $\{0\}$ and $p^i\mathbb{Z}_0^+$ are k-primary ideals, where p is a prime number and i is a positive integer. It is clear that in a semiring, all k-prime ideals are also k-primary ideals but the converse is not true in general. For instance, the primary ideals $p^i\mathbb{Z}_0^+$ are not k-prime ideals of $\mathbb{Z}_0^+$, where $i \geq 2$.

(ii) It is clear from the definition that every k-primary ideal of a semiring S is also a primary ideal of S. The converse does not hold in general. In the semiring $\mathbb{Z}_0^+$, the ideal $I = \{x \in \mathbb{Z}_0^+ \mid x \geq c\}$, where c is a fixed positive integer, is a primary ideal but not a k-primary ideal.

Now, we present a characterization theorem for k-primary ideal of a semiring in terms of the zero divisors of the corresponding quotient semiring.

Theorem 3.1 *Let S be a semiring. Then, a k-ideal P of S is k-primary if and only if the quotient semiring S/P is non-trivial and all zero divisors of S/P are nilpotent.*

Proof Let P be a k-primary ideal of S. Then, S/P is non-trivial, since $P \neq S$. Suppose that $x + P$ is a zero divisor of S/P. Therefore, $x + P \neq 0 + P$. Then, $x \notin P$ because P is a k-ideal of S. Since $x + P$ is a zero divisor, there exists $y + P \neq 0 + P$ such that $(x + P)(y + P) = 0 + P$ which implies that $xy + P = 0 + P$. So, $xy \in P$, since P is a k-ideal of S. Again, $xy \in P$, $y \notin P$ and P is a k-primary ideal of S imply that $x^n \in P$ for some $n \in \mathbb{Z}^+$. Now, $(x + P)^n = x^n + P = 0 + P$ implies that $x + P$ is nilpotent.

Conversely, let S/P be non-trivial and every zero divisor of S/P be nilpotent. Now, S/P is non-trivial means that $P \neq S$. Suppose that $x, y \in S$ be such that $xy \in P$ and $x \notin P$. Again, $xy \in P$ implies that $(x + P)(y + P) = 0 + P$. Since $x \notin P$, we have $x+P \neq 0+P$. If possible, let $y^n \notin P$ for any $n \in \mathbb{Z}^+$. Then, $y \notin P$ and so $y+P \neq 0+P$. Thus, it follows that $y + P$ is a zero divisor of S/P. Then, by our assumption, we have $y + P$ is nilpotent. Therefore, there exists $k \in \mathbb{Z}^+$ such that $(y + P)^k = 0 + P$. So $y^k + P = 0 + P$. Thus, $y^k \in P$, since P is a k-ideal of S, which is a contradiction. Hence, $y^n \in P$ for some $n \in \mathbb{Z}^+$. Consequently, P is a k-primary ideal of S.

Now, we present the definition of radical of an ideal and a fuzzy ideal of a semiring which we use extensively to characterize fuzzy k-primary ideal and hence to represent the fuzzy k-primary decomposition of a fuzzy k-ideal of a semiring.

Definition 3.3 [9] *Let I be an ideal of a semiring S. Then the radical of I, denoted by $\sqrt{I}$ is defined by $\sqrt{I} = \{a \in S \mid a^n \in I$ for some $n \in \mathbb{Z}^+\}$.*

Definition 3.4 [2] *Let S be a semiring and μ be a fuzzy ideal of S. Then, the radical of μ, denoted by $\sqrt{\mu}$ is defined as $\sqrt{\mu}(x) = \sup\limits_{n\in\mathbb{Z}^+} \mu(x^n)$ for all $x \in S$.*

The following two results regarding the radical of a fuzzy ideal play very important role in various results we prove in this paper.

Lemma 3.1 [2] *Let I be an ideal of a semiring S. Then $\sqrt{\chi_I} = \chi_{\sqrt{I}}$, where χ_I is the characteristic function of I.*

Lemma 3.2 [2] *Suppose that μ is a fuzzy ideal of a semiring S. Then,*

(i) *$\sqrt{\mu}$ is a fuzzy ideal of S;*

(ii) $\mu \subseteq \sqrt{\mu}$;

(iii) $\mu \subseteq \theta \Longrightarrow \sqrt{\mu} \subseteq \sqrt{\theta}$;

(iv) $\sqrt{\sqrt{\mu}} = \sqrt{\mu}$;

(v) $\sqrt{\mu} \cap \sqrt{\theta} = \sqrt{\mu \cap \theta} = \sqrt{\mu\theta}$.

Definition 3.5 *Let μ and θ be two fuzzy subsets of a semiring S. Then, k-intrinsic product of μ and θ, denoted by $\mu \odot_k \theta$, is defined by*

$$(\mu \odot_k \theta)(x) = \sup\Big\{ \min\Big(\mu(a_i), \mu(c_j), \theta(b_i), \theta(d_j)\Big) \mid x + \sum_{i=1}^{m} a_i b_i = \sum_{j=1}^{n} c_j d_j \Big\}.$$

Now, we are ready to present the definition of fuzzy k-primary ideals of a semiring.

Definition 3.6 *A fuzzy k-ideal μ of a semiring S is said to be a fuzzy k-primary ideal of S if it is non-constant and for any two fuzzy ideals θ and η of S, $\theta \odot_k \eta \subseteq \mu$ implies that $\theta \subseteq \mu$ or $\eta \subseteq \sqrt{\mu}$.*

Now, we proceed to set up a characterization theorem for a fuzzy k-primary ideal μ of a semiring S in terms of $Im\mu$ and μ_0, where $\mu_0 = \{x \in S \mid \mu(x) = \mu(0)\}$.

For this we have to establish the following results.

Lemma 3.3 *A k-ideal I of a semiring S is a k-primary ideal of S if and only if χ_I is a fuzzy k-primary ideal of S.*

Theorem 3.2 *If P is a k-primary ideal of a semiring S and $\alpha \in [0, 1)$, then the fuzzy subset μ of S defined by*

$$\mu(x) = \begin{cases} 1, & if\ x \in P, \\ \alpha, & otherwise \end{cases}$$

is a fuzzy k-primary ideal of S.

Proof Let $x, y \in S$. If $\min(\mu(x+y), \mu(y)) = 1$, then $x+y \in P$ and $y \in P$. This implies that $x \in P$, since P is a k-ideal of S. Therefore, $\mu(x) = 1 = \min(\mu(x+y), \mu(y))$. Again, if $\min(\mu(x + y), \mu(y)) = \alpha$, then $\mu(x) \geq \min(\mu(x + y), \mu(y))$ whence μ is a fuzzy k-ideal of S. Consider two fuzzy ideals μ_1 and μ_2 of S such that $\mu_1 \circ \mu_2 \subseteq \mu$ and $\mu_1 \nsubseteq \mu$. If possible, suppose that $\mu_2 \nsubseteq \sqrt{\mu}$. Accordingly, there exists $t \in S$ such that $\mu_2(t) > \sqrt{\mu}(t) = \sup_{n \in \mathbb{Z}^+} \mu(t^n)$. That being the case, $\sup_{n \in \mathbb{Z}^+} \mu(t^n) \neq 1$. Then, $\mu(t^n) = \alpha$ for all $n \in \mathbb{Z}^+$. Thus, it follows that $t^n \notin P$ for any $n \in \mathbb{Z}^+$. As a result, $t \notin \sqrt{P}$. Again, $\mu_1 \nsubseteq \mu$ implies that $\mu_1(z) > \mu(z)$ for some $z \in S$. Then, $\mu(z) = \alpha$ which exhibits that $z \notin P$. Since P is a k-primary ideal of S, we find that $zt \notin P$. Thus, $\mu(zt) = \alpha$. Now, $(\mu_1 \circ \mu_2)(zt) \geq \min(\mu_1(z), \mu_2(t)) > \alpha = \mu(zt)$. It demonstrates that $\mu_1 \circ \mu_2 \nsubseteq \mu$. Thus, we arrive at a contradiction. Hence, $\mu_2 \subseteq \sqrt{\mu}$. Consequently, μ is a fuzzy k-primary ideal of S.

The above theorem helps us to produce several examples of fuzzy k-primary ideals of a semiring. We provide one such example in the following.

Example 3.2 Let μ be a fuzzy ideal of the semiring $\mathbb{Z}_0^+$, defined by

$$\mu(x) = \begin{cases} 1, & \text{if } x \in p^i\mathbb{Z}_0^+,\ p \text{ prime and } \ i \in \mathbb{Z}^+, \\ 0.3, & \text{otherwise.} \end{cases}$$

Then, μ is a fuzzy k-primary ideal of $\mathbb{Z}_0^+$.

Theorem 3.3 *If μ is a fuzzy k-primary ideal of a semiring S, then* (i) $\text{Im}\mu = \{1, \alpha\}$, *where $\alpha \in [0, 1)$ and* (ii) μ_0 *is a k-primary ideal of S.*

Proof (i) Firstly, our aim is to prove $\mu(0) = 1$. If possible, let $\mu(0) < 1$. Since μ is a fuzzy ideal of S, it follows that $\mu(0) \geq \mu(x)$ for all $x \in S$. So, we have $\mu(0) > \mu(1)$ (to avoid triviality, we consider $1 \notin \mu_0$, otherwise $\mu_0 = S$). We construct two fuzzy ideals μ_1 and μ_2 of S, where

$$\mu_1(x) = \begin{cases} 1, & \text{if } x \in \mu_0, \\ 0, & \text{otherwise} \end{cases}$$

and $\mu_2(x) = \mu(0)$ for all $x \in S$. Let $x \in \mu_0$. For any $y \in S$, we have $\min(\mu_1(x), \mu_2(y)) = \mu_2(y) = \mu(0) = \mu(x) \leq \mu(xy)$, since μ is a fuzzy ideal of S. Consider another case $x \notin \mu_0$. Then, for any $y \in S$, $\min(\mu_1(x), \mu_2(y)) = 0 \leq \mu(xy)$. So we obtain that $\sup_{z=xy}\{\min(\mu_1(x), \mu_2(y))\} \leq \mu(z)$ for any $z \in S$. Thus, $(\mu_1 \circ \mu_2)(z) \leq \mu(z)$ for all $z \in S$. It demonstrates that $\mu_1 \circ \mu_2 \subseteq \mu$. Because μ is a fuzzy k-primary ideal of S, we have $\mu_1 \subseteq \mu$ or $\mu_2 \subseteq \sqrt{\mu}$. But $\mu_1(0) = 1 > \mu(0)$ and $\mu_2(1) = \mu(0) > \mu(1) = \sqrt{\mu}(1)$. Thus, we arrive at a contradiction. Consequently, $\mu(0) = 1$.

Now, our goal is to prove that $|\text{Im}\mu| = 2$. Since μ is non-constant, we have $|\text{Im}\mu| \geq 2$. If possible, let $|\text{Im}\mu| > 2$. Therefore, there exists $a \in S$ such that $1 > \mu(a) > \mu(1)$. Construct two fuzzy ideals μ_1 and μ_2 of S as follows :

$$\mu_1(x) = \begin{cases} 1, & \text{if } x \in \langle a \rangle, \\ 0, & \text{otherwise} \end{cases}$$

and $\mu_2(x) = \mu(a)$ for all $x \in S$. Suppose that $x \in \langle a \rangle$. Then, for any $y \in S$, we find that $\min(\mu_1(x), \mu_2(y)) = \mu_2(y) = \mu(a) \leq \mu(x)$ (since $x \in \langle a \rangle$) $\leq \mu(xy)$. Now, we consider $x \notin \langle a \rangle$. Then, for any $y \in S$, $\min(\mu_1(x), \mu_2(y)) = 0 \leq \mu(xy)$. Therefore, $\sup_{z=xy}\Big(\min(\mu_1(x), \mu_2(y))\Big) \leq \mu(z)$. So we find that $\mu_1 \circ \mu_2 \subseteq \mu$. Since μ is a fuzzy k-primary ideal of S, it follows that $\mu_1 \subseteq \mu$ or $\mu_2 \subseteq \sqrt{\mu}$. But $\mu_1(a) = 1 > \mu(a)$ and $\mu_2(1) = \mu(a) > \mu(1) = \sqrt{\mu}(1)$. So we arrive at a contradiction. Hence, $|\text{Im}\mu| = 2$.

(ii) We have to prove μ_0 is a k-primary ideal of S. Let A, B be two ideals of S such that $AB \subseteq \mu_0$. Then, clearly $\chi_{AB} \subseteq \chi_{\mu_0}$. So we find that $\chi_A \circ \chi_B = \chi_{AB} \subseteq \chi_{\mu_0}$. For $t \in \mu_0$, $\chi_{\mu_0}(t) = 1 = \mu(0) = \mu(t)$. Again $t \notin \mu_0$ implies that $\chi_{\mu_0}(t) = 0 \leq \mu(t)$. Therefore, $\chi_A \circ \chi_B \subseteq \chi_{\mu_0} \subseteq \mu$. Thus, we get that $\chi_A \subseteq \mu$ or $\chi_B \subseteq \sqrt{\mu}$ because μ is a fuzzy k-primary ideal of S. It can be easily checked that $\chi_A \subseteq \mu$ implies $A \subseteq \mu_0$ and $\chi_B \subseteq \sqrt{\mu}$ implies $B \subseteq \sqrt{\mu_0}$. Let $t_1 \in S$ and $t_2, t_1+t_2 \in \mu_0$. Since, μ is a fuzzy k-ideal of S, we have $\mu(t_1) \geq \min(\mu(t_1 + t_2), \mu(t_2)) = \mu(0)$. This shows that $\mu(t_1) \geq \mu(0)$. Again,

$\mu(0) \geq \mu(t_1)$, because μ is a fuzzy ideal of S which follows that $t_1 \in \mu_0$. Hence, μ_0 is a k-deal of S. Consequently, μ_0 is a k-primary ideal of S.

From Theorem 3.2 and Theorem 3.3, we have the following characterization theorem for fuzzy k-primary ideal of a semiring.

Theorem 3.4 *A fuzzy k-ideal μ of a semiring S is a fuzzy k-primary ideal of S if and only if* $\text{Im}\mu = \{1, \alpha\}$*, where* $\alpha \in [0, 1)$ *and* μ_0 *is a k-primary ideal of* S.

Now, we like to present the fuzzy version of the Theorem 3.1. For that we mention in the following, the notion of fuzzy coset determined by a fuzzy ideal of a semiring S and an element of S which was first introduced by T. K. Dutta and B. K. Biswas in [8].

Definition 3.7 [8] *Let μ be a fuzzy ideal of a semiring S and $x \in S$. Then, the fuzzy coset μ^*_x determined by μ and x is defined by* $\mu^*_x(r) = \sup\limits_{x+u=r+v} \left\{ \min(\mu(u), \mu(v)) \right\}$ *for* $r, u, v \in S$.

Theorem 3.5 [8] *Let μ be a fuzzy ideal of a semiring S. Then, the set S_μ of all fuzzy closets of μ in S is a semiring under the binary operations defined by $\mu^*_x + \mu^*_y = \mu^*_{x+y}$ and $\mu^*_x \mu^*_y = \mu^*_{xy}$ for all $x, y \in S$.*

In case of fuzzy k-ideal of a semiring, we have the following result.

Lemma 3.4 *Let S be a semiring and μ be a fuzzy k-ideal of S. Then, $S_\mu \simeq S/\mu_0$.*

Proof Let us define a function $f : S \longrightarrow S_\mu$ by $f(x) = \mu^*_x$ for all $x \in S$. For any $x, y \in S$, we have $f(x + y) = \mu^*_{x+y} = \mu^*_x + \mu^*_y = f(x) + f(y)$ and $f(xy) = \mu^*_{xy} = \mu^*_x \mu^*_y = f(x) f(y)$. Thus, '$f$' is a homomorphism. Clearly, 'f' is an epimorphism. Now, $\ker f = \{x \in S \mid f(x) = 0_{S_\mu}\} = \{x \in S \mid \mu^*_x = \mu^*_0\}$. Let $x \in \ker f$. Then, $\mu^*_x = \mu^*_0 = \mu$, since μ is a fuzzy k-ideal of S. Also, $\mu(x) \geq \sup\limits_{x+u=0+v} \{\min(\mu(x + u), \mu(u))\} = \sup\limits_{x+u=0+v} \{\min(\mu(v), \mu(u))\} = \mu^*_x(0)$. So we obtain that $\mu(x) \geq \mu^*_x(0) = \mu(0)$. Again $\mu(0) \geq \mu(x)$, since μ is a fuzzy ideal of S. It demonstrates that $x \in \ker f \Leftrightarrow \mu(x) = \mu(0) \Leftrightarrow x \in \mu_0$. Therefore, $\ker f = \mu_0$. Finally, by first isomorphism theorem of semirings, it follows that $S/\mu_0 \simeq S_\mu$.

Now, we are ready to present the fuzzy version of Theorem 3.1.

Theorem 3.6 *A fuzzy k-ideal μ of a semiring S is fuzzy k-primary if and only if S_μ is non-trivial and every zero divisor of S_μ is nilpotent.*

Proof Let μ be a fuzzy k-primary ideal of S. We have $S_\mu \simeq S/\mu_0$ by Lemma 3.4. From Theorem 3.3, it is clear that μ_0 is a k-primary ideal of S, since μ is a fuzzy k-primary ideal of S. Thus, it follows from Theorem 3.3 that each zero divisor of S/μ_0 is nilpotent. Consequently, each zero divisor of S_μ is nilpotent.

Conversely, let each zero divisor of S_μ be nilpotent. This shows that each zero divisor of S/μ_0 is nilpotent. Thus, μ_0 is a k-primary ideal of S by Theorem 3.3. Hence, μ is a fuzzy k-primary ideal of S by Theorem 3.4.

4. Fuzzy k-Irreducible Ideals of a Semiring

We start this section with the notion of fuzzy k-irreducible ideal of a semiring which is the generalization of fuzzy k-prime ideal. We show that in a Noetherian semiring S, every fuzzy k-irreducible ideal is also a fuzzy k-primary ideal of S and using this result we set up the fuzzy k-primary decomposition theorem for a fuzzy k-ideal.

Definition 4.1 *A proper k-ideal I of a semiring S is said to be k-irreducible ideal of S if for any two k-ideals J, K of S, $I = J \cap K$ implies that $I = J$ or $I = K$.*

Definition 4.2 *A non-constant fuzzy k-ideal μ of a semiring S is said to be a fuzzy k-irreducible ideal of S if for any two fuzzy k-ideals θ and η of S, $\mu = \theta \cap \eta$ implies that either $\mu = \theta$ or $\mu = \eta$.*

Now, we proceed to establish one characterization theorem for a fuzzy k-irreducible ideal of a semiring. Before that, we demonstrate following results which are useful to prove the characterization theorem.

Theorem 4.1 *Let μ be a fuzzy k-irreducible ideal of a semiring S. Then,*

(i) $1 \in \mathrm{Im}\mu$;

(ii) *there exists $\alpha \in [0, 1)$ such that $\mu(x) = \alpha$ for all $x \in S \setminus \mu_0$;*

(iii) *the ideal μ_0 is a k-irreducible ideal of S.*

Proof (i) If possible, suppose that $1 \notin \mathrm{Im}\mu$. Then, $\mu(0) < 1$. Let us define two fuzzy subsets μ_1 and μ_2 of S by

$$\mu_1(x) = \begin{cases} 1, & \text{if } x \in \mu_0, \\ \mu(x), & \text{otherwise} \end{cases}$$

and $\mu_2(x) = \mu(0)$ for all $x \in S$. It is easy to verify that μ_0 is a k-ideal of S, since μ is a fuzzy k-ideal of S. Then, using the similar method as in Theorem 3.2, we can prove that μ_1 and μ_2 are fuzzy k-ideals of S. It can be easily checked that $\mu = \mu_1 \cap \mu_2$. But $\mu \subset \mu_1$ and $\mu \subset \mu_2$. Thus, we arrive at a contradiction since μ is a fuzzy k-irreducible ideal of S. Consequently, $1 \in \mathrm{Im}\mu$.

(ii) It's sufficient to prove that the chain of level ideals is given by $\mu_0 \subset S$. If possible, let the chain of level ideals be $\mu_0 \subset \mu_{t_1} \subset S$, where $t_1 \in [0, 1)$. Then, μ is precisely given by

$$\mu(x) = \begin{cases} 1, & \text{if } x \in \mu_0, \\ t_1, & \text{if } x \in \mu_{t_1} \setminus \mu_0, \\ t_2, & \text{if } x \in S \setminus \mu_{t_1}, \end{cases}$$

where $t_2 < t_1$.

Now, let us construct two fuzzy subsets μ_3 and μ_4 as follows :

$$\mu_3(x) = \begin{cases} 1, & \text{if } x \in \mu_{t_1}, \\ \mu(x), & \text{if } x \in S \setminus \mu_{t_1} \end{cases}$$

and

$$\mu_4(x) = \begin{cases} 1, & \text{if } x \in \mu_0, \\ t_1, & \text{if } x \in \mu_{t_1} \setminus \mu_0, \\ t_3, & \text{if } x \in S \setminus \mu_{t_1}, \end{cases}$$

where $t_2 < t_3 < t_1$. It's a routine case study to check that μ_3 and μ_4 are fuzzy k-ideals of S and $\mu = \mu_3 \cap \mu_4$. But $\mu \subset \mu_3$ and $\mu \subset \mu_4$. It contradicts the fact that μ is a fuzzy k-irreducible ideal of S. Consequently, the chain of level ideals is given by $\mu_0 \subset S$ and hence μ is given by

$$\mu(x) = \begin{cases} 1, & \text{if } x \in \mu_0, \\ t_1, & \text{if } x \in S \setminus \mu_0. \end{cases}$$

(iii) Let $\mu_0 = A \cap B$ for some k-ideals A and B of S. We have $\mu_0 \subseteq A$ and $\mu_0 \subseteq B$. If possible, let $A \neq \mu_0$ and $B \neq \mu_0$. Therefore, $(A \setminus \mu_0) \cap (B \setminus \mu_0)$ is empty. Let us define two fuzzy subsets μ_5 and μ_6 as follows :

$$\mu_5(x) = \begin{cases} 1, & \text{if } x \in \mu_0, \\ t_1, & \text{if } x \in A \setminus \mu_0, \\ t_2, & \text{if } x \in S \setminus A \end{cases}$$

and

$$\mu_6(x) = \begin{cases} 1, & \text{if } x \in \mu_0, \\ t_1, & \text{if } x \in B \setminus \mu_0, \\ t_2, & \text{if } x \in S \setminus B. \end{cases}$$

Now, it's a straightforward case study to verify that μ_5 and μ_6 are fuzzy k-ideals of S and $\mu = \mu_5 \cap \mu_6$. Though we have $\mu \neq \mu_5$ and $\mu \neq \mu_6$. This contradicts the fact that μ is fuzzy k-irreducible. Consequently, $\mu_0 = A$ or $\mu_0 = B$ and hence μ_0 is k-irreducible.

Theorem 4.2 *Let I be a k-irreducible ideal of a semiring S. $\alpha \in [0, 1)$ and μ be a fuzzy ideal of S, defined as follows :*

$$\mu(x) = \begin{cases} 1, & \textit{if } x \in I, \\ \alpha, & \textit{otherwise}. \end{cases}$$

Then, μ is a fuzzy k-irreducible ideal of S.

Proof Using the similar argument as in Theorem 3.2, we can easily obtain that μ is a fuzzy k-ideal of S. Suppose that $\mu = \mu_1 \cap \mu_2$, where μ_1 and μ_2 are two fuzzy k-ideals of S, such that $\mu \subset \mu_1$ and $\mu \subset \mu_2$. Therefore, there exist $z, t \in S$ such that $\mu_1(z) > \mu(z)$ and $\mu_2(t) > \mu(t)$. Also, $z \neq t$, since $z = t$ implies that $\mu(z) < \mu_1(z)$ and $\mu(z) < \mu_2(z)$, i.e., $\mu(z) < (\mu_1 \cap \mu_2)(z)$. This contradicts the fact that $\mu = \mu_1 \cap \mu_2$. Since $\mu_1(z) > \mu(z)$ and $\mu_2(t) > \mu(t)$, we have $\mu(z) = \alpha = \mu(t)$, i.e., $z \notin I$ and $t \notin I$. Now, let us consider the k-ideals $\overline{<I, z>}$ and $\overline{<I, t>}$. Clearly, $I \subseteq \overline{<I, z>} \cap \overline{<I, t>}$. To obtain the reverse inclusion, let $a \in \overline{<I, z>} \cap \overline{<I, t>}$. This implies that $a + i_1 + s_1 z = i_2 + s_2 z$ and $a + i_3 + s_3 t = i_4 + s_4 t$, where $i_1, i_2, i_3, i_4 \in I$ and $s_1, s_2, s_3, s_4 \in S$. Again

$$\begin{aligned}
\mu_1(a) &\geq \min(\mu_1(a + i_1 + s_1 z), \mu_1(i_1 + s_1 z)) \ \ (\text{since } \mu_1 \text{ is a fuzzy } k\text{-ideal of } S) \\
&= \min(\mu_1(i_2 + s_2 z), \mu_1(i_1 + s_1 z)) \\
&\geq \min(\mu_1(s_2 z), \mu_1(s_1 z)) \ \ (\text{since } \mu_1 \text{ is a fuzzy ideal of } S) \\
&\geq \min(\mu_1(z), \mu_1(z)) = \mu_1(z) > \alpha.
\end{aligned}$$

Also,

$$\begin{aligned}
\mu_2(a) &\geq \min(\mu_2(a + i_3 + s_3 t), \mu_2(i_3 + s_3 t)) \ \ (\text{since } \mu_2 \text{ is a fuzzy } k\text{-ideal of } S) \\
&= \min(\mu_2(i_4 + s_4 t), \mu_2(i_3 + s_3 t)) \\
&\geq \min(\mu_2(s_4 t), \mu_2(s_3 t)) \ \ (\text{Since } \mu_2 \text{ is a fuzzy ideal of } S) \\
&\geq \min(\mu_2(t), \mu_2(t)) = \mu_2(t) > \alpha.
\end{aligned}$$

Therefore, $(\mu_1 \cap \mu_2)(a) > \alpha$. So $\mu(a) > \alpha$, which implies that $a \in I$. It demonstrates that $\overline{<I, z>} \cap \overline{<I, t>} \subseteq I$. Consequently, $\overline{<I, z>} \cap \overline{<I, t>} = I$. This contradicts the fact that I is a k-irreducible ideal of S, since $z \notin I$ and $t \notin I$ imply that $I \neq \overline{<I, z>}$ and $I \neq \overline{<I, t>}$ respectively. So $\mu = \mu_1$ or $\mu = \mu_2$. Consequently, μ is a fuzzy k-irreducible ideal of S.

With the help of the above theorem, we can easily produce many examples of fuzzy k-irreducible ideals of a semiring. We mention one such in the following.

Example 4.1 Consider the semiring $S = (\mathbb{N}_0, +, \cdot)$ of non-negative integers with respect to usual addition and multiplication of integers. Let μ be a fuzzy ideal of S defined as follows :

$$\mu(x) = \begin{cases} 1, & \text{if } x \in 16\mathbb{N}_0, \\ 0.4, & \text{otherwise.} \end{cases}$$

Then, by the above theorem, we can easily check that μ is a fuzzy k-irreducible ideal of S. Also, we observe that μ is not a fuzzy prime ideal of S.

Theorem 4.1 and Theorem 4.2 construct one characterization for a fuzzy k-irreducible ideal of a semiring as follows :

Theorem 4.3 *A fuzzy k-ideal μ of a semiring S is a fuzzy k-irreducible ideal of S if and only if* $\mathrm{Im}\mu = \{1, \alpha\}$, *where* $\alpha \in [0, 1)$ *and* μ_0 *is a k-irreducible ideal of S.*

The proof of the following theorem is similar to the corresponding result for the proper ideal in a Noetherian ring. So we omit this proof and refer to [5] for the proof.

Theorem 4.4 *Every proper k-ideal of a Noetherian semiring S can be expressed as a finite intersection of k-irreducible ideals.*

Now, we provide the fuzzy version of the above result.

Theorem 4.5 *Let S be a Noetherian semiring and μ be any fuzzy k-ideal of a semiring S such that* $\mathrm{Im}\mu = \{1, \alpha\}$, *where* $\alpha \in [0, 1)$. *Then, μ can be represented as finite intersection of fuzzy k-irreducible ideals of S.*

Proof Consider level ideal μ_0. Then,

$$\mu(x) = \begin{cases} 1, & \text{if } x \in \mu_0, \\ \alpha, & \text{otherwise.} \end{cases}$$

Since S is Noetherian, μ_0 can be represented as finite intersection of k-irreducible ideals, by Theorem 4.4. Suppose that $\mu_0 = A_1 \cap A_2 \cap \cdots \cap A_n$, where each A_i is a k-irreducible ideal of S. Let us define for each $1 \le i \le n$,

$$\mu_i(x) = \begin{cases} 1, & \text{if } x \in A_i, \\ \alpha, & \text{otherwise.} \end{cases}$$

Then, it can be checked that $\mu = \mu_1 \cap \mu_2 \cdots \cap \mu_n$. Also, each μ_i is fuzzy k-irreducible ideal of S, by Theorem 4.2. Hence, the proof completes.

It is important to note that every k-irreducible ideal of a semiring S may not be a k-primary ideal of S, but in a Noetherian semiring we have the following result.

Theorem 4.6 *In a Noetherian semiring S, every k-irreducible ideal of S is a k-primary ideal of S.*

Proof Let Q be a k-irreducible ideal of a Noetherian semiring S. Let $ab \in Q$ be such that $b \notin Q$. Now, we construct two ideals I and J of S as follows : $I = < a^n > + Q$ and $J = < b > + Q$. Then, clearly, $Q \subseteq I \cap J$. Let $y \in I \cap J$. Therefore, $y = a^n z + q$ for some $z \in S$ and $q \in Q$. Again $aJ \subseteq Q$ (since $ab \in Q$) and so $ay \in Q$ (since $y \in J$). Therefore, $ay = a^{n+1}z + aq$. Thus, we get that $a^{n+1}z + aq \in Q$. Also, $aq \in Q$, since $q \in Q$ and Q is an ideal of S. Thus, it follows that $a^{n+1}z \in Q$, since Q is a k-ideal of S. Construct a set $A_n = \{x \in S \mid a^n x \in Q\}$. It's easy to check that A_n is an ideal of S and $A_1 \subseteq A_2 \subseteq \cdots$ is an ascending chain of ideals. Since S is Noetherian, $A_n = A_{n+1} = \cdots$ for some $n \in \mathbb{Z}^+$. Again $a^{n+1}z \in Q$ implies that $z \in A_{n+1} = A_n$. It demonstrates that $a^n z \in Q$ which implies that $y \in Q$. Thus, $I \cap J = Q$. Since $b \notin Q$, it follows that $J \neq Q$ and hence $Q = I = < a^n > + Q$. Since Q is a k-ideal, $a^p \in Q$ for some $p \in \mathbb{Z}^+$. Consequently, Q is k-primary.

From Theorem 3.2, Theorem 4.1 and Theorem 4.6, we have the following theorem.

Theorem 4.7 Let S be a Noetherian semiring and μ be a fuzzy k-irreducible ideal of S. Then μ is also a fuzzy k-primary ideal of S.

5. Fuzzy k-Primary Decomposition of Fuzzy k-Ideal of a Semiring

In this final section, we are entering into the conclusive part of this paper which is about the fuzzy k-primary decomposition of a fuzzy k-ideal of a semiring. The following sequence of results form the base of our main desired result.

We now present the notion of fuzzy θ-primary ideal of a semiring corresponding to a fuzzy k-ideal μ of that semiring. To set up this notion we mention the following definitions and results.

Definition 5.1 [6, 7] *A fuzzy ideal μ of a semiring S is said to be a fuzzy prime ideal of S if μ is not a constant function (i.e., $|\mathrm{Im}\mu| \geq 2$) and for any two fuzzy ideals μ_1 and μ_2 of S, $\mu_1 \circ \mu_2 \subseteq \mu$ implies that either $\mu_1 \subseteq \mu$ or $\mu_2 \subseteq \mu$.*

A fuzzy prime ideal μ of S which is also a fuzzy k-ideal of S is said to be a fuzzy k-prime ideal of S.

Definition 5.2 [16] *Let X be a non-empty set and $x \in X$. Suppose that $a \in (0, 1]$. Then, a fuzzy subset x_a of X is called a fuzzy point of X if*

$$x_a(y) = \begin{cases} a, & if\ x = y, \\ 0, & otherwise. \end{cases}$$

A fuzzy point x_a is said to be contained in a fuzzy subset μ of X, denoted by $x_a \in \mu$, if $a \leq \mu(x)$.

Definition 5.3 [6] *A non-constant fuzzy ideal μ of a semiring S is said to be a fuzzy completely prime ideal of S if for any two fuzzy points x_a and y_b of S, $x_a \circ y_b \in \mu \Longrightarrow$ either $x_a \in \mu$ or $y_b \in \mu$.*

Theorem 5.1 [6] *Every fuzzy completely prime ideal of a semiring S is a fuzzy prime ideal of S.*

Lemma 5.1 *Let μ be a fuzzy k-primary ideal of a semiring S. Then, the radical $\sqrt{\mu}$ of μ is a fuzzy k-prime ideal of S.*

Proof Let μ be a fuzzy k-primary ideal of S. Suppose that x_a and y_b are two fuzzy points of S such that $x_a \circ y_b \subseteq \sqrt{\mu}$ and $x_a \notin \sqrt{\mu}$. Then, it follows that

$$\begin{aligned}
&(xy)_{\min(a,b)} \in \sqrt{\mu} \\
&\Rightarrow \min(a, b) \leq \sqrt{\mu}(xy) \\
&\Rightarrow \min(a, b) \leq \sup_n \mu(xy)^n \\
&\Rightarrow \min(a, b) \leq \sup_n \mu(x^n y^n) \quad \text{(since } S \text{ is commutative)} \\
&\Rightarrow \min(a, b) \leq \mu(x^n y^n) \quad \text{for all } n \\
&\Rightarrow (x^n y^n)_{\min(a,b)} \in \mu \quad \text{for all } n \\
&\Rightarrow x_a^n \circ y_b^n \in \mu \quad \text{for all } n.
\end{aligned}$$

But $x_a^n \notin \mu$ (since $\mu \subseteq \sqrt{\mu}$ and $x_a \notin \sqrt{\mu}$). Thus, it follows that $y_b^n \in \sqrt{\mu}$, since μ is a fuzzy k-primary ideal of S. Consequently, $b \leq \sup_k \mu(y^n)^k = \sup_m \mu(y^m) = \sqrt{\mu}(y)$.

Therefore, $y_b \in \sqrt{\mu}$. This shows that $\sqrt{\mu}$ is a fuzzy completely prime ideal of S and hence a fuzzy prime ideal of S, by Theorem 5.1. Also, μ is a fuzzy k-ideal of S, since it is fuzzy k-primary. Consequently, $\sqrt{\mu}$ is a fuzzy k-prime ideal of S.

Definition 5.4 *Let μ be a fuzzy k-primary ideal and θ be a fuzzy prime ideal of a semiring S. Then, μ is said to be a fuzzy θ-primary ideal of S if $\sqrt{\mu} = \theta$.*

By using Lemma 3.2 (iv), the following result can be easily deduced.

Proposition 5.1 *Let $\alpha_1, \alpha_2, \cdots, \alpha_m$ be fuzzy θ-primary ideals of a semiring S. Then, $\bigcap\limits_{i=1}^{m} \alpha_i$ is a fuzzy θ-primary ideal of S.*

In the following, we introduce the concept of fuzzy k-primary decomposition of a fuzzy k-ideal of a semiring.

Definition 5.5 *Let μ be a fuzzy k-ideal of a semiring S. Then, μ is said to have a fuzzy k-primary decomposition if μ can be expressed as $\mu = \bigcap\limits_{i=1}^{n} \mu_i$, where each μ_i is a fuzzy k-primary ideal of S.*

A fuzzy k-primary decomposition $\mu = \bigcap\limits_{i=1}^{n} \mu_i$ is said to be reduced if $\bigcap\limits_{\substack{j=1 \\ j\neq i}}^{n} \mu_j \nsubseteq \mu_i$ and $\sqrt{\mu_i}$'s are all distinct for $i = 1, \cdots, n$.

From Theorem 4.5 and Theorem 4.7, we have the following result.

Theorem 5.2 *Let S be a Noetherian semiring and μ be any fuzzy k-ideal of S such that* $\mathrm{Im}\mu = \{1, \alpha\}$, *where* $\alpha \in [0, 1)$. *Then, μ can be represented as finite intersection of fuzzy k-primary ideals of S, that is μ has a fuzzy k-primary decomposition.*

Theorem 5.3 *Let μ be a fuzzy k-ideal of a semiring S. If μ has a fuzzy k-primary decomposition, then μ also has a reduced fuzzy k-primary decomposition.*

Proof Let μ be a fuzzy k-ideal of a semiring S and $\mu = \bigcap\limits_{i=1}^{n} \mu_i$ be a fuzzy k-primary decomposition of μ. Suppose that $\{\mu_1, \mu_2, \cdots, \mu_n\}$ consists of k_1 number of fuzzy θ_1-primary ideals, k_2 number of fuzzy θ_2-primary ideals, $\cdots$, k_m number of fuzzy θ_m-primary ideals of S. So, we have $k_1 + k_2 + \cdots + k_m = n$. Now, suppose that $\bigcap\limits_{j=1}^{k_i} \mu_{ij} = \alpha_i$, with $i = 1, \cdots, m$, where μ_{ij}'s are fuzzy θ_i-primary ideals of S. From Proposition , it follows that α_i's are also fuzzy k-primary ideals of S. So $\mu = \bigcap\limits_{i=1}^{m} \alpha_i$ is a fuzzy k-primary decomposition of μ, where α_i's are distinct for all $i = 1, \cdots, m$. Now, we discard α_i from the set $\{\alpha_1, \alpha_2, \cdots, \alpha_m\}$ if $\bigcap\limits_{\substack{j=1 \\ j\neq i}}^{m} \alpha_j \subseteq \alpha_i$. We continue this process until $\bigcap\limits_{\substack{j=1 \\ j\neq i}}^{m} \alpha_j \nsubseteq \alpha_i$ for all $i = 1, \cdots, m$. Suppose that $\alpha_{11}, \alpha_{12}, \cdots, \alpha_{1p}$ are discarded. Then, $\mu = \bigcap\limits_{i=1}^{m-p} \beta_i$, where $\{\beta_1, \beta_2, \cdots, \beta_{m-p}\} = \{\alpha_1, \alpha_2, \cdots, \alpha_m\} \setminus \{\alpha_{11}, \alpha_{12}, \cdots, \alpha_{1p}\}$. It is clear that $\mu = \bigcap\limits_{i=1}^{m-p} \beta_i$ is a reduced fuzzy k-primary decomposition of μ.

In the following, we introduce the notion of fuzzy colon ideal in semiring which helps us to prove the uniqueness of the reduced fuzzy k-primary decomposition of fuzzy k-ideal of a semiring.

Definition 5.6 *Let μ and θ be two fuzzy subsets of a semiring S. Then, we define the fuzzy colon ideal $(\mu : \theta)$ of S by $(\mu : \theta)(x) = \sup\limits_{\lambda \in IFS(S)} \{\lambda(x) \mid \lambda \circ \theta \subseteq \mu\}$, where $IFS(S)$ denotes the set of all fuzzy subsets of S.*

Lemma 5.2 *Let μ, θ, η be three fuzzy ideals of a semiring S. Then, we have the following*

(i) *$\mu \subseteq (\mu : \theta)$ and $(\mu : \theta) \circ \theta \subseteq \mu$ and $((\mu : \theta) : \eta) = (\mu : (\theta \circ \eta))$.*

(ii) *If $\mu \subseteq \theta$, then $(\mu : \eta) \subseteq (\theta : \eta)$ and $(\eta : \theta) \subseteq (\eta : \mu)$.*

(iii) *If $\mu_1, \mu_2, \cdots, \mu_n$ are fuzzy ideals of S, then $(\bigcap\limits_{i=1}^{n} \mu_i : \mu) = \bigcap\limits_{i=1}^{n} (\mu_i : \mu)$.*

Proof Let $x \in S$.

(i) We have $(\mu : \theta)(x) = \sup\limits_{\lambda \in IFS(S)} \{\lambda(x) \mid \lambda \circ \theta \subseteq \mu\} \geq \mu(x)$ since μ is a fuzzy ideal of S imply that $\lambda \circ \mu \subseteq \chi_S \circ \mu \subseteq \mu$. Therefore, $\mu \subseteq (\mu : \theta)$.

$$\begin{aligned}
&((\mu : \theta) \circ \theta)(x) \\
&\quad = \sup\{\min((\mu : \theta)(a), \theta(b)) \mid x = ab \text{ for some } a, b \in S\} \\
&\quad = \sup\{\min\Big(\sup_{\lambda \in IFS(S)} \{\lambda(a) \mid \lambda \circ \theta \subseteq \mu\}, \theta(b)\Big) \mid x = ab \text{ for some } a, b \in S\} \\
&\quad \leq \sup\{\min(\lambda(a), \theta(b)) \mid x = ab \text{ for some } a, b \in S\} \\
&\quad = (\lambda \circ \theta)(x) \leq \mu(x).
\end{aligned}$$

Thus, $(\mu : \theta) \circ \theta \subseteq \mu$.

$((\mu : \theta) : \eta)(x) = \sup\limits_{\lambda \in IFS(S)} \{\lambda(x) \mid \lambda \circ \eta \subseteq (\mu : \theta)\}$. Again $\lambda \circ \eta \subseteq \mu : \theta \Leftrightarrow (\lambda \circ \eta) \circ \theta \subseteq \mu \Leftrightarrow \lambda \circ (\eta \circ \theta) \subseteq \mu \Leftrightarrow \lambda \circ (\theta \circ \eta) \subseteq \mu$ (since S is commutative). Thus,

$$\begin{aligned}
((\mu : \theta) : \eta)(x) &= \sup_{\lambda \in IFS(S)} \{\lambda(x) \mid \lambda \circ \eta \subseteq \mu : \theta\} \\
&= \sup_{\lambda \in IFS(S)} \{\lambda(x) \mid \lambda \circ (\theta \circ \eta) \subseteq \mu\} \\
&= (\mu : (\theta \circ \eta))(x).
\end{aligned}$$

Therefore, $((\mu : \theta) : \eta) = (\mu : (\theta \circ \eta))$.

(ii)

$$\begin{aligned}
(\mu : \eta)(x) &= \sup_{\lambda \in IFS(S)} \{\lambda(x) \mid \lambda \circ \eta \subseteq \mu\} \\
&\leq \sup_{\lambda \in IFS(S)} \{\lambda(x) \mid \lambda \circ \eta \subseteq \theta\} = (\theta : \eta)(x).
\end{aligned}$$

Consequently, $(\mu : \eta) \subseteq (\theta : \eta)$. Again

$$
\begin{aligned}
(\eta : \theta)(x) &= \sup_{\lambda \in IFS(S)} \{\lambda(x) \mid \lambda \circ \theta \subseteq \eta\} \\
&\leq \sup_{\lambda \in IFS(S)} \{\lambda(x) \mid \lambda \circ \mu \subseteq \eta\} = (\eta : \mu)(x).
\end{aligned}
$$

Thus, $(\eta : \theta) \subseteq (\eta : \mu)$.

(iii)

$$
\begin{aligned}
(\bigcap_{i=1}^{n}(\mu_i : \mu))(x) &= \inf_{1 \leq i \leq n} \{(\mu_i : \mu)(x)\} \\
&= \inf_{1 \leq i \leq n} \{ \sup_{\lambda \in IFS(S)} \{\lambda(x) \mid \lambda \circ \mu \subseteq \mu_i\}\} \\
&= \sup_{\lambda \in IFS(S)} \{\lambda(x) \mid \lambda \circ \mu \subseteq \bigcap_{i=1}^{n} \mu_i\} \\
&= (\bigcap_{i=1}^{n} \mu_i : \mu).
\end{aligned}
$$

We conclude our paper by proving the uniqueness of reduced fuzzy k-primary decomposition of a fuzzy k-ideal in semiring.

Theorem 5.4 *Let μ be a fuzzy k-ideal of a semiring S and $\mu = \bigcap_{i=1}^{n} \mu_i$ be a reduced fuzzy k-primary decomposition of μ. Let λ be a fuzzy k-prime ideal of S. Then, $\lambda = \sqrt{\mu_i}$ for some $i = 1, 2, \cdots, n$ if and only if there exists a fuzzy k-ideal θ of S such that $\theta \not\subseteq \mu$ and $\sqrt{(\mu : \theta)} = \lambda$, i.e., $(\mu : \theta)$ is a fuzzy λ-primary ideal of S.*

Proof Suppose that $\lambda = \sqrt{\mu_i}$ for some $i = 1, 2, \cdots, n$. Since $\mu = \bigcap_{i=1}^{n} \mu_i$ is a reduced fuzzy k-primary decomposition, $\bigcap_{\substack{j=1 \\ i \neq j}}^{n} \mu_j \not\subseteq \mu_i$. Then, there exists $t_i \in S$ such that $(\bigcap_{\substack{j=1 \\ i \neq j}}^{n} \mu_j)(t_i) > \mu_i(t_i)$. Consider $(\bigcap_{\substack{j=1 \\ i \neq j}}^{n} \mu_j)(t_i) = \alpha$, where $\alpha \in (0, 1]$. Now, we construct a fuzzy subset θ of S as follows :

$$
\theta(x) = \begin{cases} \alpha, & \text{if } x \in \overline{< t_i >}, \\ 0, & \text{otherwise.} \end{cases}
$$

Then, by a simple case study and using the fact that $\overline{< t_i >}$ is a k-ideal of S, we can prove that θ is a fuzzy k-ideal of S. Now, $\theta(t_i) = \alpha = (\bigcap_{\substack{j=1 \\ i \neq j}}^{n} \mu_j)(t_i) > \mu_i(t_i) > (\bigcap_{i=1}^{n} \mu_i)(t_i) = \mu(t_i)$. Therefore, $\theta \not\subseteq \mu$. So we obtain that $(\mu : \theta) = ((\bigcap_{i=1}^{n} \mu_i) : \theta) = \bigcap_{i=1}^{n}(\mu_i : \theta)$ by Lemma 5.2 (iii). Now, by Lemma 3.2 (iv), it follows that $\sqrt{(\mu : \theta)} = \sqrt{\bigcap_{i=1}^{n}(\mu_i : \theta)} =$

$\bigcap\limits_{i=1}^{n} \sqrt{(\mu_i : \theta)}$.

$$\begin{aligned}\sqrt{(\mu_i : \theta)}(x) &= \sup_n\{(\mu_i : \theta)(x^n)\}\\ &= \sup_n\{\sup_{\sigma \in IFS(S)}\{\sigma(x^n) \mid \sigma \circ \theta \subseteq \mu_i\}\}\\ &\leq \sup_n\{\sqrt{\mu_i}(x^n)\}\\ &= \sqrt{\sqrt{\mu_i}}(x) = \sqrt{\mu_i}(x) \text{ (by Lemma 3.2).}\end{aligned}$$

Thus, we find that $\sqrt{(\mu_i : \theta)} \subseteq \sqrt{\mu_i}$. Again we have $\mu_i \subseteq (\mu_i : \theta)$ by Lemma 5.2 (i). Therefore, $\sqrt{\mu_i} \subseteq \sqrt{(\mu_i : \theta)}$. So we find that $\sqrt{\mu_i} = \sqrt{(\mu_i : \theta)}$. Thus, we get that

$$\sqrt{\bigcap_{i=1}^{n}(\mu_i : \theta)} = \bigcap_{i=1}^{n} \sqrt{(\mu_i : \theta)} = \bigcap_{i=1}^{n} \sqrt{\mu_i} = \bigcap_{i=1}^{n} \lambda = \lambda.$$

Consequently, $\sqrt{(\mu : \theta)} = \lambda$.

Conversely, suppose that there exists a fuzzy k-ideal θ of S such that $\theta \nsubseteq \mu$ and $\sqrt{(\mu : \theta)} = \lambda$. Now, $\theta \nsubseteq \mu$ implies that $\theta \nsubseteq \bigcap\limits_{i=1}^{n} \mu_i$. Then, there exists $j \in \{1, 2, \cdots, n\}$ such that $\theta \nsubseteq \mu_j$. Again $\sqrt{(\mu : \theta)} = \lambda$ implies that $\bigcap\limits_{i=1}^{n} \sqrt{(\mu_i : \theta)} = \lambda$. So $\lambda \subseteq \sqrt{(\mu_i : \theta)}$ for all $i = 1, 2, \cdots, n$. Then, $\lambda \subseteq \sqrt{(\mu_j : \theta)}$. Again, by Lemma 5.2 (i), it follows that $(\mu_j : \theta) \circ \theta \subseteq \mu_j$. Since μ_j is a fuzzy k-primary ideal of S and $\theta \nsubseteq \mu_j$, it follows that $(\mu_j : \theta) \subseteq \sqrt{\mu_j}$. This implies that $\sqrt{(\mu_j : \theta)} \subseteq \sqrt{\mu_j}$, by Lemma 3.2. Also, it is clear from Lemma 5.2 (i), that $\sqrt{\mu_j} \subseteq \sqrt{(\mu_j : \theta)}$. Thus, $\sqrt{\mu_j} = \sqrt{(\mu_j : \theta)}$. So we have $\lambda \subseteq \sqrt{\mu_j}$. Now, $\bigcap\limits_{i=1}^{n} \sqrt{(\mu_j : \theta)} = \lambda \Longrightarrow \sqrt{\mu_j} \cap \bigcap\limits_{\substack{k=1\\k \neq j}}^{n} \sqrt{(\mu_k : \theta)} = \lambda \Longrightarrow \sqrt{\mu_j} \circ \bigcap\limits_{\substack{k=1\\k \neq j}}^{n} \sqrt{(\mu_k : \theta)} = \lambda$. But $\lambda = \bigcap\limits_{i=1}^{n} \sqrt{(\mu_i : \theta)} \subset \bigcap\limits_{\substack{k=1\\k \neq j}}^{n} \sqrt{(\mu_k : \theta)}$ and λ is a fuzzy k-prime ideal of S. Therefore, $\sqrt{\mu_j} \subseteq \lambda$. Consequently, we obtain that $\lambda = \sqrt{\mu_j}$.

Finally, we have the following result.

Theorem 5.5 *Let μ be a fuzzy k-ideal of a semiring S and $\mu = \bigcap\limits_{i=1}^{n} \mu_i$ with $\sqrt{\mu_i} = \lambda_i$ for $i = 1, 2, \cdots n$ and $\mu = \bigcap\limits_{i=1}^{m} \theta_i$ with $\sqrt{\theta_i} = \gamma_i$ for $i = 1, 2, \cdots, m$ be two reduced fuzzy k-primary decompositions of μ. Then, $n = m$ and $\{\lambda_1, \lambda_2, \cdots, \lambda_n\} = \{\gamma_1, \gamma_2, \cdots, \gamma_m\}$.*

Proof Let $\lambda_i \in \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$. Then, from Theorem 5.4, it follows that there exists $\theta \nsubseteq \mu$ such that $\sqrt{(\mu : \theta)} = \lambda_i$. So there exists $j \in \{1, 2, \cdots, m\}$ such that $\lambda_i = \sqrt{\theta_j} = \gamma_j \in \{\gamma_1, \gamma_2, \cdots, \gamma_m\}$. Thus, $\{\lambda_1, \lambda_2, \cdots, \lambda_n\} \subseteq \{\gamma_1, \gamma_2, \cdots, \gamma_m\}$. Similarly, we can prove $\{\gamma_1, \gamma_2, \cdots, \gamma_m\} \subseteq \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$. Consequently, $\{\lambda_1, \lambda_2, \cdots, \lambda_n\} = \{\gamma_1, \gamma_2, \cdots, \gamma_m\}$ and hence $m = n$.

6. Conclusion

The main idea of the present paper is to establish the Lasker-Noether theorem for a commutative semiring which leads us to define the fuzzy k-primary decomposition of a fuzzy k-ideal. We show by an example that an arbitrary ideal in a Noetherian semiring may not be expressed as finite intersection of primary ideals of that semiring. But we are able to prove that if we consider an arbitrary k-ideal of a commutative Noetherian semiring, then it can be expressed as finite intersection of primary k-ideals. We are also able to prove the fuzzy version of the above result that is in a commutative Noetherian semiring, every fuzzy k-ideal can be decomposed uniquely as finite intersection of fuzzy k-primary ideals of that semiring. In a similar fashion, one can think of extending the primary decomposition theorem in some other algebraic structures in spite of a commutative semiring. From that point of view, our work on primary decomposition theorem in a commutative Noetherian semiring set up a new horizon and surely it develops the study of some further research.

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